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# EXISTENCE OF NONTRIVIAL SOLUTIONS FOR SEMILINEAR PROBLEMS WITH STRICTLY DIFFERENTIABLE NONLINEARITY

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The existence of a nontrivial solution for semilinear elliptic problems with strictly differentiable nonlinearity is proved. A result of homological linking under nonstandard geometrical assumption is also shown. Techniques of Morse theory are employed.

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## 1. Introduction

Since the paper of Amann and Zehnder [1], the existence of nontrivial solutions  $u$  for semilinear elliptic problems of the form

$$-\Delta u = g(u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad (1.1)$$

with  $g(0) = 0$ , has been the object of several studies, in which topological and variational methods are successfully applied. We refer the reader to [2, 3, 8, 10]. In particular, since the combination of linking theorems and Morse theory has turned out to be very fruitful, it is customary to impose conditions on  $g$  that guarantee that the associated functional  $f : H_0^1(\Omega) \rightarrow \mathbb{R}$ , given by

$$f(u) = \frac{1}{2} \int_{\Omega} |Du|^2 dx - \int_{\Omega} G(u) dx, \quad G(s) = \int_0^s g(t) dt, \quad (1.2)$$

is of class  $C^2$ .

In a recent paper [12], Perera and Schechter have proved a result of Amann-Zehnder type under assumptions that imply  $f$  to be only of class  $C^1$ . More precisely, about the regularity of  $g$ , they assume that  $g$  is continuous, there exist in  $\mathbb{R}$  the limits

$$\lim_{s \rightarrow -\infty} \frac{g(s)}{s}, \quad \lim_{s \rightarrow +\infty} \frac{g(s)}{s}, \quad \lim_{s \rightarrow 0} \frac{g(s)}{s} \quad (1.3)$$

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and that

$$\frac{g(s)}{s} \text{ is Lipschitz continuous in a neighbourhood of } 0. \quad (1.4)$$

One could observe that hypothesis (1.4) allows  $f$  not to be of class  $C^2$ , but it does not include every  $g$  satisfying the usual assumption that  $g$  is of class  $C^1$  and  $g'$  is bounded. In particular, condition (1.4) is not stable if we add to  $g$  a term of the form

$$\frac{|s|^{3/2}}{1+s^2}. \quad (1.5)$$

The first purpose of this paper is to extend the result of [12] in such a way that also the classical smooth case is included. Our result is the following.

**THEOREM 1.1.** *Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^n$  and  $g : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function satisfying  $g(0) = 0$  and*

(a) *there exists  $C \geq 0$  such that*

$$|g(s)| \leq C(1 + |s|); \quad (1.6)$$

(b) *there exists  $\alpha \in \mathbb{R}$  such that*

$$\lim_{s \rightarrow \pm\infty} \frac{g(s)}{s} = \alpha. \quad (1.7)$$

If we denote by  $(\lambda_m)$  the sequence of the eigenvalues of  $-\Delta$  with homogeneous Dirichlet boundary condition, let us assume that  $\alpha \neq \lambda_m$  for any  $m \in \mathbb{N}$ . Moreover, let us suppose that  $g$  is strictly differentiable at 0 (see Definition 3.1 below) and that there exists  $m \in \mathbb{N}$  with either  $g'(0) < \lambda_m < \alpha$  or  $g'(0) > \lambda_m > \alpha$ .

Then (1.1) admits a nontrivial solution.

Theorem 1.1 is in fact a particular case of a more general result, which will be presented in Section 2.

**Remark 1.2.** If, as in [12], we have  $g(s) = \gamma s(s)$ , with  $\gamma$  Lipschitz continuous in a neighbourhood of 0, then it is easy to see that  $g$  is strictly differentiable at 0.

A second purpose of the paper is to improve the saddle theorem proved in [11, Theorem 1.4], also mentioned in [12], in which the functional is of class  $C^2$ , but nonstandard geometrical assumptions are considered. We will prove the following.

**THEOREM 1.3.** *Let  $H$  be a Hilbert space such that  $H = H_- \oplus H_+$  with  $\dim H_- < \infty$  and  $H_+$  closed in  $H$ . Let  $f : H \rightarrow \mathbb{R}$  be a functional of class  $C^2$  and assume that*

$$c_0 = \inf_{H_+} f > -\infty, \quad c_1 = \sup_{H_-} f < +\infty, \quad (1.8)$$

*$f$  satisfies  $(PS)_c$  for every  $c \in [c_0, c_1]$ ,  $f''(u)$  is a Fredholm operator at every critical point  $u$  in  $f^{-1}([c_0, c_1])$ .*

*Then there exists a critical point  $u$  of  $f$  with  $c_0 \leq f(u) \leq c_1$  and  $m(f, u) \leq \dim H_- \leq m^*(f, u)$ .*

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In [11] it is only shown that there exist critical points  $\underline{u}$ ,  $\bar{u}$  with  $c_0 \leq f(\bar{u}) \leq f(\underline{u}) \leq c_1$  and  $m(f, \underline{u}) \leq \dim H_- \leq m^*(f, \bar{u})$ , but one cannot say if there exists a critical point  $u = \underline{u} = \bar{u}$ , as in the case with standard geometrical assumptions (see [8]), or not. Our improvement is related to the fact that, according to Proposition 4.3 below, also under the nonstandard geometrical assumptions of Theorem 1.3, it is possible to recognize a homological linking structure.

The paper is organized as follows: in Section 2 we state the result of existence of nontrivial solutions; Sections 3 and 4 are devoted to prove some auxiliary results, while in Section 5 we prove the main theorems.

## 2. Existence of a nontrivial solution

Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^n$  and  $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  be a Carathéodory function satisfying

- (g<sub>0</sub>)  $g(x, 0) = 0$  for a.e.  $x \in \Omega$ ;
- (g<sub>1</sub>) there exists  $C \geq 0$  such that  $|g(x, s)| \leq C(1 + |s|)$  for a.e.  $x \in \Omega$  and every  $s \in \mathbb{R}$ ;
- (g<sub>2</sub>) for a.e.  $x \in \Omega$ , the function  $\{s \mapsto g(x, s)\}$  is strictly differentiable at 0 (see Definition 3.1 below) with  $D_s g(\cdot, 0) \in L^\infty(\Omega)$ ;
- (g<sub>3</sub>) there exist  $\hat{C} \geq 0$  and  $\delta > 0$  such that, for a.e.  $x \in \Omega$ , we have

$$\forall s, t \in ]-\delta, \delta[ : |g(x, s) - g(x, t)| \leq \hat{C}|s - t|. \quad (2.1)$$

If we set  $G(x, s) = \int_0^s g(x, t) dt$ , it is well known that the functional  $f : H_0^1(\Omega) \rightarrow \mathbb{R}$  defined by

$$f(u) = \frac{1}{2} \int_{\Omega} |Du|^2 dx - \int_{\Omega} G(x, u) dx \quad (2.2)$$

is of class  $C^1$ .

We denote by  $m(f, 0)$  the supremum of the dimensions of the linear subspaces of  $H_0^1(\Omega)$  where the quadratic form

$$Q(u) = \int_{\Omega} |Du|^2 dx - \int_{\Omega} D_s g(x, 0) u^2 dx \quad (2.3)$$

is negative definite, and by  $m^*(f, 0)$  the supremum of the dimensions of the linear subspaces of  $H_0^1(\Omega)$  where  $Q$  is negative semidefinite. We call  $m(f, 0)$  (resp.,  $m^*(f, 0)$ ) the strict (resp., large) Morse index of  $f$  at 0.

**THEOREM 2.1.** *Assume that  $H_0^1(\Omega) = X_- \oplus X_+$  with  $\dim X_- < \infty$  and  $X_+$  closed in  $H_0^1(\Omega)$ . Suppose also that*

$$c_0 = \inf_{X_+} f > -\infty, \quad c_1 = \sup_{X_-} f < +\infty, \quad (2.4)$$

and that  $f$  satisfies (PS)<sub>c</sub> for every  $c \in [c_0, c_1]$ ,

If it is  $\dim X_- \notin [m(f, 0), m^*(f, 0)]$ , then the problem

$$-\Delta u = g(x, u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad (2.5)$$

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We changed “(g0)” to “(g<sub>0</sub>).” Please check similar cases throughout.

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admits a nontrivial solution  $u$ .

*Remark 2.2.* Under the assumption of **Theorem 1.1**, it is well known that  $f$  satisfies  $(PS)_c$  for any  $c \in \mathbb{R}$  and the geometrical assumptions of **Theorem 2.1**. Since it is clear that also  $(g_0)-(g_3)$  are satisfied, **Theorem 1.1** is a consequence of **Theorem 2.1**.

### 3. Computations of critical groups

*Definition 3.1.* Let  $\Phi$  be a map from an open subset  $U$  of a normed space  $X$  to a normed space  $Y$  and let  $u \in U$ . We say that  $\Phi$  is *strictly differentiable* at  $u$  (*strongly differentiable* in the sense of [6]), if there exists a continuous linear map  $L : X \rightarrow Y$  such that

$$\lim_{\substack{(w_1, w_2) \rightarrow (u, u) \\ w_1 \neq w_2}} \frac{\Phi(w_1) - \Phi(w_2) - L(w_1 - w_2)}{\|w_1 - w_2\|} = 0. \quad (3.1)$$

Of course, in such a case  $\Phi$  is Fréchet differentiable at  $u$  and  $L = \Phi'(u)$ .

*Definition 3.2.* Let  $\mathbb{K}$  be a field,  $X$  be a metric space and  $f : X \rightarrow \mathbb{R}$  be a continuous function. For  $u \in X$  and  $c = f(u)$ , let us set

$$\forall q \in \mathbb{Z} : C_q(f, u) = H_q(f^c, f^c \setminus \{u\}), \quad (3.2)$$

where  $f^c = \{v \in X : f(v) \leq c\}$  and  $H_q(A, B)$  denotes the  $q$ th singular homology group of the pair  $(A, B)$ , with coefficients in  $\mathbb{K}$  (see, e.g., [14]). The vector space  $C_q(f, u)$  is called *the  $q$ th critical group* of  $f$  at  $u$ . Because of the excision property, we may replace  $f$  by  $f|_U$  for any neighborhood  $U$  of  $u$  in  $X$ .

*Definition 3.3.* Let  $X$  be a Banach space,  $U$  an open subset of  $X$  and  $f : U \rightarrow \mathbb{R}$  be a function of class  $C^1$ . Let  $C$  be a closed subset of  $X$  with  $C \subseteq U$ . We say that  $f$  satisfies *the Palais-Smale condition* ( $(PS)$ , for short) *on  $C$* , if every sequence  $(u_h)$  in  $C$  with  $f(u_h)$  bounded and  $f'(u_h) \rightarrow 0$  admits a convergent subsequence. In the case  $C = A = X$ , we simply say that  $f$  satisfies  $(PS)$ .

Let  $c \in \mathbb{R}$ . We say that  $f$  satisfies *the Palais-Smale condition at level  $c$*  ( $(PS)_c$ , for short), if every sequence  $(u_h)$  in  $U$  with  $f(u_h) \rightarrow c$  and  $f'(u_h) \rightarrow 0$  admits a convergent subsequence.

Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^n$  ( $n \geq 3$ ),  $1 \leq p < (n+2)/(n-2)$  and  $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  be a Carathéodory function satisfying

( $g_1$ ) there exists  $C \geq 0$  such that  $|g(x, s)| \leq C(1 + |s|^p)$  for a.e.  $x \in \Omega$  and every  $s \in \mathbb{R}$ .  
Let  $u_0 \in H_0^1(\Omega)$  be an isolated weak solution of the semilinear problem

$$-\Delta u = g(x, u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \quad (3.3)$$

By regularity theory, we automatically have  $u_0 \in L^\infty(\Omega)$ . Moreover, let us assume that:

( $g_2$ ) for a.e.  $x \in \Omega$ , the function  $\{s \mapsto g(x, s)\}$  is strictly differentiable at  $u_0(x)$  and  $D_s g(\cdot, u_0) \in L^\infty(\Omega)$ ;

( $g'_3$ ) there exist  $\hat{C} \geq 0$  and  $\delta > 0$  such that for a.e.  $x \in \Omega$

$$\forall s, t \in ]-\delta, \delta[: |g(x, u_0(x) + s) - g(x, u_0(x) + t)| \leq \hat{C}|s - t|. \quad (3.4)$$

Let  $f : H_0^1(\Omega) \rightarrow \mathbb{R}$  be the functional

$$f(u) = \frac{1}{2} \int_{\Omega} |Du|^2 dx - \int_{\Omega} G(x, u) dx, \quad (3.5)$$

where  $G(x, s) = \int_0^s g(x, t) dt$ , and let  $Q : H_0^1(\Omega) \rightarrow \mathbb{R}$  be the quadratic form

$$Q(u) = \int_{\Omega} |Du|^2 dx - \int_{\Omega} D_s g(x, u_0) u^2 dx. \quad (3.6)$$

Finally, let  $m(f, u_0)$  and  $m^*(f, u_0)$  be defined as in [Section 2](#).

**THEOREM 3.4.** *We have that  $C_q(f, u_0) = \{0\}$  for every  $q \leq m(f, u_0) - 1$  and every  $q \geq m^*(f, u_0) + 1$ .*

The proof will be given at the end of the section.

As a first step, we approximate the functional  $f$  with suitable functionals  $f_{\lambda}$  of class  $C^1$  with  $f'_{\lambda}$  strictly differentiable at  $u_0$  and such that the critical groups of  $f_{\lambda}$  at  $u_0$  are independent of  $\lambda$ .

Let us denote by  $\|\cdot\|_q$  the norm of  $L^q(\Omega)$  and by  $\|\cdot\|_{1,2}$  the norm of  $H_0^1(\Omega)$ .

**Remark 3.5.** Up to substitute  $g$  with  $\tilde{g} : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$\tilde{g}(x, s) = g(x, u_0(x) + s) - g(x, u_0(x)), \quad (3.7)$$

we may assume that  $u_0 = 0$  and that  $g(x, 0) = 0$ .

**LEMMA 3.6.** *There exists a constant  $\bar{C} > 0$  such that, for a.e.  $x \in \Omega$  and for any  $s \in \mathbb{R}$ , we have*

$$|g(x, s)| \leq \bar{C}(1 + |s|^{p-1})|s|. \quad (3.8)$$

*Proof.* If  $0 < |s| < \delta$ , then by ( $g'_3$ ) it is

$$\left| \frac{g(x, s)}{s} \right| \leq \hat{C}. \quad (3.9)$$

Otherwise, if  $|s| \geq \delta$ , then it is

$$\left| \frac{g(x, s)}{s} \right| \leq \frac{C(1 + |s|^p)}{|s|} \leq \frac{C}{\delta} + C|s|^{p-1}. \quad (3.10)$$

Hence the assertion follows.  $\square$

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Now let  $\delta > 0$  be as in  $(g'_3)$  and  $\vartheta \in C_c^\infty(\mathbb{R})$  such that  $0 \leq \vartheta \leq 1$ ,  $\text{supt}(\vartheta) \subseteq ]-\delta, \delta[$  and

$$\begin{aligned} \vartheta(s) &= 1 \quad \text{if } s \in \left[-\frac{\delta}{4}, \frac{\delta}{4}\right], \\ 0 \leq \vartheta \leq \frac{1}{2} \quad &\text{if } s \in [-\delta, \delta] \setminus \left[-\frac{\delta}{2}, \frac{\delta}{2}\right]. \end{aligned} \quad (3.11)$$

For every  $\lambda \in [0, 1]$  let us define  $g_\lambda(x, s) = g(x, \vartheta(\lambda s)s)$  and let  $f_\lambda : H_0^1(\Omega) \rightarrow \mathbb{R}$  be the functional

$$f_\lambda(u) = \frac{1}{2} \int_\Omega |Du|^2 dx - \int_\Omega G_\lambda(x, u) dx, \quad (3.12)$$

where  $G_\lambda(x, s) = \int_0^s g_\lambda(x, t) dt$ . It is clear that:

- (a) for every  $\lambda > 0$  and for a.e.  $x \in \Omega$ , the function  $\{s \mapsto g_\lambda(x, s)\}$  is Lipschitz continuous uniformly with respect to  $x$ ;
- (b) for every  $\lambda$  and for a.e.  $x \in \Omega$ , the function  $\{s \mapsto g_\lambda(x, s)\}$  is strictly differentiable at 0 with  $D_s g_\lambda(x, 0) = D_s g(x, 0)$ ;
- (c) for a.e.  $x \in \Omega$ , the functions  $\{(\lambda, s) \mapsto g_\lambda(x, s)\}$  and  $\{(\lambda, s) \mapsto G_\lambda(x, s)\}$  are continuous;
- (d) there exists  $\bar{C} \geq 0$  such that  $|g_\lambda(x, s)| \leq \bar{C}(1 + |s|^p)$ ,  $|G_\lambda(x, s)| \leq \bar{C}(1 + |s|^{p+1})$ .

**THEOREM 3.7.** *The following facts hold:*

- (i) *for every  $\lambda \in [0, 1]$ , the functional  $f_\lambda$  is of class  $C^1$ ;*
- (ii) *there exists an open bounded neighbourhood  $U$  of 0 in  $H_0^1(\Omega)$  such that, for every  $\lambda \in [0, 1]$ , 0 is the only critical point of  $f_\lambda$  in  $\bar{U}$ ;*
- (iii) *for every  $\lambda \in ]0, 1]$ ,  $f'_\lambda$  is strictly differentiable at 0 with  $\langle f''_\lambda(0)v, v \rangle = Q(v)$ .*

*Proof.* It is readily seen that assertion (i) holds.

Let us consider assertion (ii). By contradiction, let us assume that there exist  $(\lambda_h)$  in  $[0, 1]$  and  $(u_h)$  in  $H_0^1(\Omega)$  with  $u_h \neq 0$  and  $u_h \rightarrow 0$  strongly in  $H_0^1(\Omega)$  such that  $f'_{\lambda_h}(u_h) = 0$ . Up to a subsequence,  $\lambda_h \rightarrow \lambda$  in  $[0, 1]$ . Since  $u_h$  is a critical point of  $f_{\lambda_h}$ , we have that  $u_h$  is a weak solution of

$$-\Delta u = g_{\lambda_h}(x, u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \quad (3.13)$$

Let

$$a_h = \begin{cases} \frac{g_{\lambda_h}(x, u_h)}{u_h} & \text{where } u_h \neq 0, \\ 0 & \text{where } u_h = 0. \end{cases} \quad (3.14)$$

By **Lemma 3.6** it is

$$|a_h| \leq \left| \frac{g_{\lambda_h}(x, u_h)}{u_h} \right| = \left| \frac{g(x, \vartheta(\lambda_h u_h)u_h)}{u_h} \right| \leq \bar{C}(1 + |\vartheta(\lambda_h u_h)u_h|^{p-1}) \leq \bar{C}(1 + |u_h|^{p-1}). \quad (3.15)$$

Since  $u_h$  is bounded in  $L^{2n/(n-2)}(\Omega)$ , then  $a_h$  belongs to  $L^q(\Omega)$  with  $q > n/2$  and

$$\|a_h\|_q \leq C' \left(1 + \|u_h\|_{2n/(n-2)}^{p-1}\right) \leq M. \quad (3.16)$$

Hence  $u_h$  is a weak solution of the linear problem

$$-\Delta u = a_h u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \quad (3.17)$$

By [7, Theorem 3.13.1]  $u_h \in L^\infty(\Omega)$  and there exists  $C > 0$  such that  $\|u_h\|_\infty \leq C\|Du_h\|_2$ . Hence  $u_h \rightarrow 0$  in  $L^\infty(\Omega)$ . Since  $\vartheta = 1$  on  $[-\delta/4, \delta/4]$ , for  $h$  sufficiently large we have that  $u_h$  is a weak solution of (3.3). It follows that 0 is not an isolated solution of (3.3): a contradiction.

Finally, let us consider assertion (iii). Let  $L : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$  be the continuous linear operator such that

$$\langle Lv, w \rangle = \langle Lw, v \rangle, \quad \langle Lv, v \rangle = Q(v). \quad (3.18)$$

Let  $(u_h), (v_h), (w_h)$  in  $H_0^1(\Omega)$  be such that  $u_h \rightarrow 0$ ,  $w_h \rightarrow 0$  in  $H_0^1(\Omega)$  and  $\|v_h\|_{1,2} \leq 1$ . Up to a subsequence,  $w_h \rightarrow 0$  and  $u_h \rightarrow 0$  a.e. in  $\Omega$ . We have that

$$\begin{aligned} & \left| \langle f'_\lambda(w_h), v_h \rangle - \langle f'_\lambda(u_h), v_h \rangle - \langle L(w_h - u_h), v_h \rangle \right| \\ &= \left| \int_{\{x \in \Omega : w_h(x) \neq u_h(x)\}} \left[ \frac{g_\lambda(x, w_h) - g_\lambda(x, u_h)}{w_h - u_h} - D_s g(x, 0) \right] (w_h - u_h) v_h dx \right| \\ &\leq C \left( \int_{\{x \in \Omega : w_h(x) \neq u_h(x)\}} \left| \frac{g_\lambda(x, w_h) - g_\lambda(x, u_h)}{w_h - u_h} - D_s g(x, 0) \right|^{n/2} dx \right)^{2/n} \\ &\quad \times \|w_h - u_h\|_{1,2} \|v_h\|_{1,2}. \end{aligned} \quad (3.19)$$

Then it is

$$\begin{aligned} & \frac{\left| \langle f'_\lambda(w_h), v_h \rangle - \langle f'_\lambda(u_h), v_h \rangle - \langle L(w_h - u_h), v_h \rangle \right|}{\|w_h - u_h\|_{1,2}} \\ &\leq C \left( \int_{\{x \in \Omega : w_h(x) \neq u_h(x)\}} \left| \frac{g_\lambda(x, w_h) - g_\lambda(x, u_h)}{w_h - u_h} - D_s g(x, 0) \right|^{n/2} dx \right)^{2/n} \|v_h\|_{1,2} \\ &\leq C \left( \int_\Omega \left| \frac{g_\lambda(x, w_h) - g_\lambda(x, u_h)}{w_h - u_h} - D_s g(x, 0) \right|^{n/2} \chi_{\{x \in \Omega : w_h(x) \neq u_h(x)\}} dx \right)^{2/n}. \end{aligned} \quad (3.20)$$

By (a) and (b) we can apply Lebesgue's theorem, obtaining

$$\left( \int_\Omega \left| \frac{g_\lambda(x, w_h) - g_\lambda(x, u_h)}{w_h - u_h} - D_s g(x, 0) \right|^{n/2} \chi_{\{x \in \Omega : w_h(x) \neq u_h(x)\}} dx \right)^{2/n} \rightarrow 0. \quad (3.21)$$

Therefore

$$\lim_{h \rightarrow +\infty} \frac{\langle f'_\lambda(w_h), v_h \rangle - \langle f'_\lambda(u_h), v_h \rangle - \langle L(w_h - u_h), v_h \rangle}{\|w_h - u_h\|_{1,2}} = 0 \quad (3.22)$$



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and assertion (iii) follows.  $\square$

**THEOREM 3.8.** *The critical groups  $C_q(f_\lambda, 0)$  are independent of  $\lambda$ . In particular*

$$\forall q \in \mathbb{Z} : C_q(f, 0) \approx C_q(f_1, 0). \quad (3.23)$$

*Proof.* Let  $U$  be an open bounded neighbourhood of 0 in  $H_0^1(\Omega)$  as in assertion (ii) of **Theorem 3.7**. We claim that if  $\lambda_h \rightarrow \lambda$  in  $[0, 1]$ , then  $\|f_{\lambda_h}|_{\overline{U}} - f_\lambda|_{\overline{U}}\|_{1,\infty} \rightarrow 0$ . Let  $(u_h)$  be a sequence in  $\overline{U}$ . Up to a subsequence,  $u_h \rightharpoonup u$  in  $H_0^1(\Omega)$  and  $u_h \rightarrow u$  a.e. in  $\Omega$ . It is

$$\begin{aligned} f_{\lambda_h}(u_h) - f_\lambda(u_h) &= \int_{\Omega} [G_{\lambda_h}(x, u_h) - G_\lambda(x, u_h)] dx \\ &= \int_{\Omega} [G_{\lambda_h}(x, u_h) - G_\lambda(x, u)] dx + \int_{\Omega} [G_\lambda(x, u) - G_\lambda(x, u_h)] dx. \end{aligned} \quad (3.24)$$

By (c), (d) and Lebesgue's theorem we deduce that

$$\int_{\Omega} [G_{\lambda_h}(x, u_h) - G_\lambda(x, u)] dx \rightarrow 0. \quad (3.25)$$

Therefore  $f_{\lambda_h} \rightarrow f_\lambda$  uniformly on  $\overline{U}$ .

Now, let  $v_h \in H_0^1(\Omega)$  with  $\|v_h\|_{1,2} \leq 1$ . Up to a subsequence  $v_h \rightharpoonup v$  in  $H_0^1(\Omega)$ ,  $v_h \rightarrow v$  in  $L^{2n/(n-2)}(\Omega)$  and  $v_h \rightarrow v$  a.e. in  $\Omega$ . It is

$$\begin{aligned} &|\langle f'_{\lambda_h}(u_h), v_h \rangle - \langle f'_\lambda(u_h), v_h \rangle| \\ &= \left| \int_{\Omega} [g_{\lambda_h}(x, u_h) - g_\lambda(x, u_h)] v_h dx \right| \\ &= \left| \int_{\Omega} [g(x, \vartheta(\lambda_h u_h) u_h) - g(x, \vartheta(\lambda u_h) u_h)] v_h dx \right| \\ &\leq C \left( \int_{\Omega} |g(x, \vartheta(\lambda_h u_h) u_h) - g(x, \vartheta(\lambda u_h) u_h)|^{2n/(n+2)} dx \right)^{(n+2)/2n} \|v_h\|_{1,2}. \end{aligned} \quad (3.26)$$

As before we have that

$$\int_{\Omega} |g_{\lambda_h}(x, u_h) - g_\lambda(x, u_h)|^{2n/(n+2)} dx \rightarrow 0. \quad (3.27)$$

It follows that  $f'_{\lambda_h} \rightarrow f'_\lambda$  uniformly on  $\overline{U}$ . Finally, since  $U$  is bounded and  $g$  has subcritical growth, we have that for every  $\lambda \in [0, 1]$   $f_\lambda$  satisfies (PS) in  $\overline{U}$ . By [5, Theorem 5.2] the assertion follows.  $\square$

In the second part of this section we deduce from [6] a generalization of the classical Shifting theorem (see [3, Theorem I.5.4], [10, Theorem 8.4]).

Let  $H$  be a Hilbert space,  $U$  be an open subset of  $H$ ,  $u_0 \in U$  and  $f : U \rightarrow \mathbb{R}$  be a function of class  $C^1$  such that  $f'$  is strictly differentiable at  $u_0$  and  $f''(u_0)$  is a Fredholm operator. In particular,  $f'$  is Lipschitz continuous in a neighbourhood of  $u_0$ . Let  $L : H \rightarrow H$

be the linear operator defined by

$$\forall v, w \in H : \langle Lv, w \rangle = \langle f''(u_0)v, w \rangle, \quad (3.28)$$

let  $V_0 = \ker L$  and let  $P_{V_0}$  be the orthogonal projection on  $V_0$ . We also denote by  $m(f, u_0)$  (resp.,  $m^*(f, u_0)$ ) the strict (resp., large) Morse index of  $f$  at  $u_0$ .

**THEOREM 3.9.** *Let  $u_0$  be an isolated critical point of  $f$ . Then there exist a neighbourhood  $\hat{U}$  of  $P_{V_0}u_0$  in  $V_0$  and a function  $\hat{f} : \hat{U} \rightarrow \mathbb{R}$  of class  $C^1$  with locally Lipschitz gradient such that  $P_{V_0}u_0$  is an isolated critical point of  $\hat{f}$  and*

$$\forall q \in \mathbb{Z} : C_q(f, u_0) \approx \begin{cases} C_{q-m(f, u_0)}(\hat{f}, P_{V_0}u_0) & \text{if } m(f, u_0) < \infty, \\ \{0\} & \text{if } m(f, u_0) = \infty, \end{cases} \quad (3.29)$$

$$\begin{aligned} \forall q \leq m(f, u_0) - 1 : C_q(f, u_0) &= \{0\}, \\ \forall q \geq m^*(f, u_0) + 1 : C_q(f, u_0) &= \{0\}. \end{aligned} \quad (3.30)$$

*Proof.* Without loss of generality, we may assume that  $u_0 = 0$ . From [6, Theorem 1.2] we also see that the generalized Morse lemma holds also in this setting. Arguing as in the proof of [10, Theorem 8.4], we find that (3.29) holds. Actually, in our case  $f$  is of class  $C^{2-0}$  instead of  $C^2$ , but the proof of [10, Theorem 8.4] remains valid also in this case.

On the other hand, also the proof of [10, Theorem 8.5] can be easily adapted from the  $C^2$  to the  $C^{2-0}$  case. Therefore we have that  $C_q(\hat{f}, P_{V_0}u_0) = \{0\}$  if  $q \geq \dim V_0 + 1$ . Since  $m^*(f, u_0) = m(f, u_0) + \dim V_0$ , the other assertions follow from (3.29).  $\square$

Finally, let us prove **Theorem 3.4**.

*Proof.* By **Remark 3.5** we may assume that  $u_0 = 0$ . Let  $f_\lambda : H_0^1(\Omega) \rightarrow \mathbb{R}$  be as in (3.12). By **Theorem 3.7** we have that  $f_1$  is of class  $C^1$  with  $f_1'$  strictly differentiable at 0 and 0 is an isolated critical point of  $f_1$ . Moreover,  $f_1''(0)$  is a Fredholm operator. By **Theorem 3.8** it is

$$\forall q \in \mathbb{Z} : C_q(f, 0) \approx C_q(f_1, 0). \quad (3.31)$$

On the other hand, since  $Q(u) = \langle f_1''(0)u, u \rangle$ , we have that  $m(f, 0) = m(f_1, 0)$  and  $m^*(f, 0) = m^*(f_1, 0)$ . From **Theorem 3.9** the assertion follows.  $\square$

#### 4. Homological linking

Throughout this section,  $X$  will denote a Banach space,  $B_r(u)$  the open ball of center  $u \in X$  and radius  $r$  and  $f : X \rightarrow \mathbb{R}$  a function of class  $C^1$ . We set  $K = \{u \in X : f'(u) = 0\}$  and, for every  $c \in \mathbb{R}$ ,

$$K_c = \{u \in X : f'(u) = 0, f(u) = c\}. \quad (4.1)$$

We also denote by  $H_*$  singular homology.

First of all, let us recall from [4] an extension of the homological linking of [3].

**Definition 4.1.** Let  $D, S, A$  be three subsets of  $X$ ,  $m \in \mathbb{N}$  and  $\mathbb{K}$  a field. We say that  $(D, S)$  links  $A$  homologically in dimension  $m$  (over  $\mathbb{K}$ ), if  $S \subseteq D$ ,  $S \cap A = \emptyset$  and there exists  $z \in H_m(X, S; \mathbb{K})$  belonging to the image of  $i_* : H_m(D, S; \mathbb{K}) \rightarrow H_m(X, S; \mathbb{K})$  but not of  $j_* : H_m(X \setminus A, S; \mathbb{K}) \rightarrow H_m(X, S; \mathbb{K})$ , where  $i : (D, S) \rightarrow (X, S)$  and  $j : (X \setminus A, S) \rightarrow (X, S)$  are the inclusion maps.

It is clear that, if  $(D, S)$  links  $A$  homologically, then  $D \cap A \neq \emptyset$ .

**THEOREM 4.2.** Let  $D, S, A$  be three subsets of  $X$  such that  $(D, S)$  links  $A$  homologically in dimension  $m$  and let  $z \in H_m(X, S; \mathbb{K})$  be as in **Definition 4.1**. Assume that

$$\inf_A f > -\infty, \quad \sup_D f < +\infty, \quad \forall u \in S : f(u) < \inf_A f \quad (4.2)$$

and define

$$c = \inf \{ b \in \mathbb{R} : S \subseteq f^b \text{ and } z \text{ belongs to the image of the} \\ \text{homomorphism induced by inclusion } H_m(f^b, S; \mathbb{K}) \longrightarrow H_m(X, S; \mathbb{K}) \}. \quad (4.3)$$

Suppose that  $f$  satisfies (PS) and that each element of  $K_c$  is isolated in  $K$ .

Then  $\inf_A f \leq c \leq \sup_D f$  and there exists  $u \in K_c$  with  $C_m(f, u) \neq \{0\}$ .

To prove our main results we need the following.

**PROPOSITION 4.3.** Let  $X = X_- \oplus X_+$ , with  $\dim X_- < \infty$  and  $X_+$  closed in  $X$ . Assume that

$$c_0 = \inf_{X_+} f > -\infty, \quad c_1 = \sup_{X_-} f < +\infty \quad (4.4)$$

and that  $f$  satisfies  $(PS)_c$  for every  $c \in [c_0, c_1]$ .

Then there exists a compact pair  $(D, S)$  in  $X$  such that

$$\max_D f \leq c_1, \quad \forall u \in S : f(u) < c_0 \quad (4.5)$$

and such that  $(D, S)$  links  $X_+$  homologically in dimension  $\dim X_-$  over all  $\mathbb{K}$ .

*Proof.* Since  $f$  satisfies  $(PS)_c$  for every  $c \in [c_0, c_1]$ , there exists  $r > 0$  such that  $K \cap f^{-1}([c_0, c_1]) \subseteq (B_r(0) \cap X_-) \oplus X_+$ . Moreover, there exist  $\delta, \sigma > 0$  such that

$$\|P_{X_-} u\| \geq r, \\ c_0 - \delta \leq f(u) \leq c_1 + \delta \implies \|f'(u)\| > \sigma, \quad (4.6)$$

where  $P_{X_-}$  denotes the projection on  $X_-$  induced by the decomposition  $X = X_- \oplus X_+$ . Let  $c > 0$  be such that  $\|P_{X_-} u\| \leq c\|u\|$  for any  $u \in X$  and let

$$R = c \frac{c_1 - c_0 + \delta}{\sigma} + r + \delta, \quad \rho_1 = 1, \quad \rho_2 = R - r - \delta, \\ C = X \setminus [(B_{r+\rho_1+\rho_2}(0) \cap X_-) \oplus X_+]. \quad (4.7)$$

By [5, Theorem 2.1] applied to the function  $f|_{\{u \in X : f(u) \geq c_0 - \delta\}}$ , there exist a continuous function

$$\tau : \overline{B_{\rho_1}(C)} \cap \{u \in X : c_0 - \delta \leq f(u) < c_1 + \delta\} \longrightarrow [0, +\infty) \quad (4.8)$$

and a continuous map

$$\eta : \left( \overline{B_{\rho_1}(C)} \cap \{u \in X : c_0 - \delta \leq f(u) < c_1 + \delta\} \right) \times [0, 1] \longrightarrow \{u \in X : f(u) \geq c_0 - \delta\} \quad (4.9)$$

such that

- (a)  $\tau(u) = 0 \Leftrightarrow f(u) = c_0 - \delta$ ;
- (b)  $\|\eta(u, t) - u\| \leq \tau(u)t$ ;
- (c)  $f(\eta(u, t)) \leq f(u) - \sigma\tau(u)t$ ;
- (d)  $f(\eta(u, 1)) = c_0 - \delta$ .

Let  $\vartheta_1 : \mathbb{R} \rightarrow [0, 1]$  be a continuous function such that

$$\vartheta_1(s) = 1 \quad \text{if } s \leq c_1, \quad \vartheta_1(s) = 0 \quad \text{if } s \geq c_1 + \delta/2, \quad (4.10)$$

and let  $\vartheta_2 : X \rightarrow [0, 1]$  be a continuous function such that

$$\vartheta_2(u) = 1 \quad \text{if } \|u\| \geq R, \quad \vartheta_2(u) = 0 \quad \text{if } \|u\| \leq R - \delta. \quad (4.11)$$

Let  $\mathcal{H} : X \times [0, 1] \rightarrow X$  be the deformation defined by

$$\mathcal{H}(u, t) = \begin{cases} \eta(u, \vartheta_1(f(u))\vartheta_2(P_{X_-}u)t) & \text{if } u \in \overline{B_{\rho_1}(C)}, c_0 - \delta \leq f(u) \leq c_1 + \delta, \\ u & \text{if } f(u) \leq c_0 - \delta, \\ u & \text{if } f(u) \geq c_1 + \delta/2, \\ u & \text{if } \|P_{X_-}u\| \leq R - \delta. \end{cases} \quad (4.12)$$

If  $u \in X_-$ , we have that

$$\|P_{X_-}\mathcal{H}(u, t) - u\| \leq c\|\mathcal{H}(u, t) - u\| \leq c \frac{f(u) - f(\mathcal{H}(u, t))}{\sigma} \leq c \frac{c_1 - c_0 + \delta}{\sigma} < R - r. \quad (4.13)$$

It follows

$$\begin{aligned} \|P_{X_-}u\| \leq r &\implies \mathcal{H}(u, t) = u, \\ u \in X_-, &\implies f(\mathcal{H}(u, 1)) < c_0, \\ \|u\| \geq R &\implies \|P_{X_-}(\mathcal{H}(u, t))\| \geq r, \quad \forall t \in [0, 1]. \end{aligned} \quad (4.14)$$

It is clear that  $(X, (X_- \setminus B_r(0)) \oplus X_+)$  links  $X_+$  homologically in dimension  $\dim X_-$  and that the inclusion map

$$i : \left( \overline{B_R(0)} \cap X_-, \partial B_R(0) \cap X_- \right) \longrightarrow (X, (X_- \setminus B_r(0)) \oplus X_+) \quad (4.15)$$

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induces an isomorphism in homology. Let  $m = \dim X_-$  and

$$B = \overline{B_R(0)} \cap X_-, \quad E = \partial B_R(0) \cap X_-, \quad F = (X_- \setminus B_r(0)) \oplus X_+. \quad (4.16)$$

Consider now the commutative diagram

$$\begin{array}{ccccc} H_m(B, E) & \longrightarrow & H_m(X, E) & \longleftarrow & H_m(X \setminus X_+, E) \\ \downarrow & & \downarrow & & \downarrow \\ H_m(X, F) & \xrightarrow{Id} & H_m(X, F) & \longleftarrow & H_m(X \setminus X_+, F) \end{array} \quad (4.17)$$

where horizontal rows are induced by the inclusions and the vertical rows are isomorphisms. We have that there exists  $z \in H_m(X, E)$  belonging to the image of  $H_m(B, E) \rightarrow H_m(X, E)$  such that  $i_*(z) \in H_m(X, F)$ , but not to the image of  $H_m(X \setminus X_+, F) \rightarrow H_m(X, F)$ . Let us consider the compact sets  $D = \mathcal{H}(B, 1)$  and  $S = \mathcal{H}(E, 1)$ . We have that

$$\max_D f \leq c_1, \quad \max_S f < c_0, \quad S \subseteq F. \quad (4.18)$$

Consider now the commutative diagram

$$\begin{array}{ccccc} H_m(B, E) & \longrightarrow & H_m(X, E) & & \\ \mathcal{H}_*(\cdot, 1) \downarrow & & \mathcal{H}_*(\cdot, 1) \downarrow & & \\ H_m(D, S) & \longrightarrow & H_m(X, S) & \longleftarrow & H_m(X \setminus X_+, S) \\ \downarrow & & \downarrow & & \downarrow \\ H_m(X, F) & \xrightarrow{Id} & H_m(X, F) & \longleftarrow & H_m(X \setminus X_+, F) \end{array} \quad (4.19)$$

Since  $\mathcal{H}(\cdot, 1) : (X, E) \rightarrow (X, F)$  is homotopically equivalent to the identity map, then  $(D, S)$  links  $X_+$  homologically in dimension  $m = \dim X_-$  and the assertions follows.  $\square$

## 5. Proof of the main results

*proof of Theorem 2.1.* By contradiction, let us assume that 0 is the unique solution of (2.5). Since  $m = \dim X_- \notin [m(f, 0), m^*(f, 0)]$ , by Theorem 3.4 it is  $C_m(f, 0) = \{0\}$ . By Proposition 4.3 there exists a compact pair  $(D, S)$  in  $H_0^1(\Omega)$  such that

$$\forall u \in S : f(u) < \inf_{X_+} f \quad (5.1)$$

and  $(D, S)$  links  $X_+$  homologically in dimension  $m$  over all  $\mathbb{K}$ . By Theorem 4.2 there exists a critical point  $u \in H_0^1(\Omega)$  of  $f$  such that  $C_m(f, u) \neq \{0\}$ . Hence  $u \neq 0$  and  $u$  is a weak solution of (2.5): a contradiction.  $\square$

*proof of Theorem 1.3.* Let  $(D, S)$  be as in Proposition 4.3. By [13, Proposition 3.9 and Remark] there exists  $\delta > 0$  such that  $f$  satisfies  $(PS)_c$  for every  $c \in [c_0 - \delta, c_1 + \delta]$  and

$f''(u)$  is a Fredholm operator at every critical point  $u$  in  $f^{-1}([c_0 - \delta, c_1 + \delta])$ . Let us argue by contradiction and set

$$\begin{aligned} K_1 &= \{u \in H : c_0 - \delta \leq f(u) \leq c_1 + \delta, f'(u) = 0, m^*(f, u) < \dim H_-\}, \\ K_2 &= \{u \in H : c_0 - \delta \leq f(u) \leq c_1 + \delta, f'(u) = 0, m(f, u) > \dim H_-\}. \end{aligned} \quad (5.2)$$

Then  $K_1, K_2$  are two disjoint compact sets whose union is the critical set of  $f$  in  $f^{-1}([c_0 - \delta, c_1 + \delta])$ . By Marino-Prodi perturbation lemma [9, Teorema 2.2], there exists a functional  $\hat{f} : H \rightarrow \mathbb{R}$  of class  $C^2$  such that

$$\inf_{H_+} \hat{f} > c_0 - \delta/2, \quad \sup_{H_-} \hat{f} < c_1 + \delta/2, \quad \max_S \hat{f} < \inf_{H_+} \hat{f}, \quad (5.3)$$

$\hat{f}$  satisfies  $(PS)_c$  for every  $c \in [c_0 - \delta/2, c_1 + \delta/2]$ ,  $\hat{f}$  has only non-degenerate critical points  $u$  in  $\hat{f}^{-1}([c_0 - \delta/2, c_1 + \delta/2])$ , with either  $m(\hat{f}, u) < \dim H_-$  or  $m^*(\hat{f}, u) > \dim H_-$ . If we apply **Theorem 4.2** to  $\hat{f}$ , we find a critical point  $u$  of  $\hat{f}$  with  $c_0 - \delta/2 \leq \hat{f}(u) \leq c_1 + \delta/2$  and  $C_m(\hat{f}, u) \neq \{0\}$ , where  $m = \dim H_-$ . By the Morse lemma, we have  $m(\hat{f}, u) = m$  and a contradiction follows.  $\square$

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